

## 5 Relations and Partitions

Ch.22 of PJE.

In the last Section we defined congruence modulo  $m$ , showing that it satisfied the three properties of being reflexive, symmetric and transitive. We then defined congruence classes, using congruence modulo  $m$ , showing that the classes satisfied the properties of being disjoint and the union of them containing every integer.

We now generalize the ideas of congruences and congruence classes of  $\mathbb{Z}$  to *any* set  $X$ .

### 5.1 Relations

**Definition 5.1.1** A **relation** on a non-empty set  $X$  is a non-empty subset  $\mathcal{R} \subseteq X \times X$ , i.e. a collection of ordered pairs.

If  $(a, b) \in \mathcal{R}$  we say that  $a$  **is related to**  $b$  and write  $aRb$  or  $a \sim b$ .

If  $(a, b) \notin \mathcal{R}$  we say that  $a$  is **not** related to  $b$  and write  $a \not\sim b$  (we can also write  $aNRb$  but it isn't often used).

**Note** we have *two* ways of writing a relation, either as  $\mathcal{R}$ , a set of ordered pairs, or using  $\sim$ . We will use both notations.

**Example 5.1.2** (i) Three different relations on  $\mathbb{Z}$  could be

a)  $x < y$ , in which case  $\mathcal{R} = \{ \dots, (1, 2), (1, 19), (-3, 0), \dots \}$ ,

b)  $x = y$  in which case  $\mathcal{R} = \{ \dots, (1, 1), (100, 100), (-3, -3), \dots \}$ ,

c)  $x \equiv y \pmod{7}$  in which case  $\mathcal{R} = \{ \dots, (1, 8), (8, 1), (-15, 6), (21, 0), \dots \}$ .

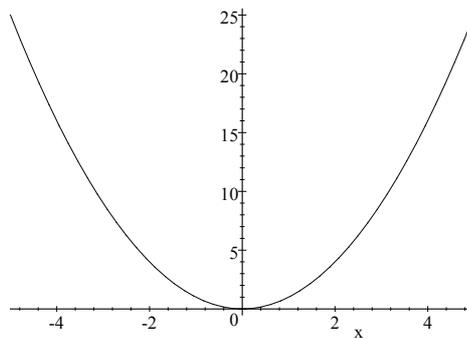
(ii) If  $A = \{a, b, c, d, e, f\}$  then

$$\begin{aligned} \mathcal{R} = \{ & (a, a), (a, b), (b, a), (b, b), (c, c), (c, e), \\ & (e, c), (e, e), (d, d), (f, f) \} \end{aligned}$$

is a relation on  $A$ .

(iii)  $\mathcal{R} = \{(x, x^2) : x \in \mathbb{R}\}$  is a relation on  $\mathbb{R}$ .

This relation is also the graph of the function  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$ . This can be represented pictorially as



So  $\mathcal{R}$  is represented by all the points on the parabola, a subset of the Cartesian plane.

In fact, the graph of any function  $f : X \rightarrow X$ , defined as

$$G_f = \{(x, f(x)) : x \in X\} \subseteq X \times X,$$

is a relation. But the converse is not true, not all relations are graphs of functions.

**Example 5.1.3** If  $X = \{1, 2, 3\}$  then

- a)  $\mathcal{R}_1 = \{(1, 1), (2, 2)\}$  is **not** the graph of a function since 3 is not related to anything, i.e. it has no image,
- b)  $\mathcal{R}_2 = \{(1, 1), (2, 3), (1, 2), (3, 2)\}$  is **not** a graph of a function since 1 has two images.

## 5.2 Equivalence Relations

**Definition 5.2.1** Suppose that  $\sim$  is a relation on a set  $X$ . Then

- i)  $\sim$  is **reflexive** if  $\forall a \in X, a \sim a$ ,
- ii)  $\sim$  is **symmetric** if  $\forall a, b \in X, a \sim b \Rightarrow b \sim a$ ,
- iii)  $\sim$  is **transitive** if  $\forall a, b, c \in X, a \sim b$  and  $b \sim c \Rightarrow a \sim c$ .

If  $\sim$  satisfies **all** three parts then we say that  $\sim$  is an **equivalence relation**.

**Note** that in (ii) or (iii) the elements  $a, b \in X$  or  $a, b, c \in X$  need **not** be different.

**Example 5.2.2** Let  $\sim$  on  $\mathbb{R}$  be given by

$$\forall a, b \in \mathbb{R}, a \sim b \text{ iff } a < b,$$

the **order relation** on  $\mathbb{R}$ .

Is **not** reflexive since  $1 \not\sim 1$ ,

Is **not** symmetric since  $1 < 2$  but  $2 \not< 1$ ,

Is **transitive**, since if  $a < b$  and  $b < c$  then  $a < c$ .

**Aside** if a property does not hold give a *counterexample*, if it does hold try to give a proof. For example,  $a < b$ , and  $b < c$  means  $0 < b - a$  and  $0 < c - b$ . Add these together using the fact that the sum of two positive numbers is positive to get  $0 < b - a + c - b = c - a$ , which implies  $a < c$ . **End of aside**

**Example 5.2.3** Let  $X = \{1, 2, 3\}$ .

(i)  $\mathcal{R}_1 = \{(1, 2), (2, 1), (3, 3)\}$ .

Is **not** reflexive since  $(1, 1) \notin \mathcal{R}_1$ ,

Is symmetric. (Check that if  $(a, b) \in \mathcal{R}_1$  then  $(b, a) \in \mathcal{R}_1$ . There are three checks to be made.)

Is **not** transitive since  $(1, 2), (2, 1) \in \mathcal{R}_1$  but  $(1, 1) \notin \mathcal{R}_1$  (so  $a = 1, b = 2$  and  $c = 1$  in the definition of transitive, highlighting the point above that  $a, b, c$  need **not** be different).

(ii)  $\mathcal{R}_2 = \{(1, 1), (2, 2), (3, 3)\}$ .

Is reflexive,

Is symmetric,

Is transitive.

Hence  $\mathcal{R}_2$  is an equivalence relation.

**Important** Equivalence relations on a general set  $X$  are generalisations of congruences on  $\mathbb{Z}$ .

We have already seen that for  $m \geq 1$  then  $x \sim y$  defined by  $x \equiv y \pmod{m}$  is an equivalence relation on  $\mathbb{Z}$ .

We can give some examples of equivalence relations not based on congruences in  $\mathbb{Z}$ .

**Example 5.2.4** Let  $X = \mathbb{Z}$  and  $\sim$  be given by  $x \sim y$  if, and only if,  $3x + 8y$  is divisible by 11. Show that  $\sim$  is an equivalence relation.

**Solution**

*Reflexive* For any  $x \in \mathbb{Z}$  we have  $3x + 8x = 11x$  which is divisible by 11, so  $x \sim x$ .

*Symmetric:* Let  $x, y \in \mathbb{Z}$  and assume  $x \sim y$ . Then

$$3x + 8y \text{ is divisible by 11, i.e. } 3x + 8y = 11t$$

for some  $t \in \mathbb{Z}$ . Note that

$$\begin{aligned} 3y + 8x &= 11y + 11x - 8y - 3x \\ &= 11(y + x) - (3x + 8y) \\ &= 11(y + x - t). \end{aligned}$$

Hence  $3y + 8x$  is divisible by 11 and so  $y \sim x$ .

*Transitive:* Let  $x, y, z \in \mathbb{Z}$  and assume  $x \sim y$ ,  $y \sim z$ . Then

$$3x + 8y = 11n \quad \text{while} \quad 3y + 8z = 11m,$$

for some integers  $n$  and  $m$ . Add these equations together to get

$$3x + 11y + 8z = 11n + 11m,$$

that is

$$3x + 8z = 11(n + m - y).$$

Hence  $3x + 8z$  is divisible by 11 and so  $y \sim z$ . ■

**Example 5.2.5** Let  $X = \mathbb{Z}$  and  $\sim$  be given by  $x \sim y$  if, and only if,  $(x - y)(x + y)$  is divisible by 7. Show that  $\sim$  is an equivalence relation.

**Solution**

*Reflexive:* For any  $x \in \mathbb{Z}$  we have  $(x - x)(x + x) = 0$  which is divisible by 7, so  $x \sim x$ .

*Symmetric:* Let  $x, y \in \mathbb{Z}$  and assume  $x \sim y$ . Then

$$(x - y)(x + y) = 7n,$$

for some  $n \in \mathbb{Z}$ . Thus

$$(y - x)(y + x) = 7(-n)$$

is divisible by 7 and hence  $y \sim x$ .

*Transitive:* Let  $x, y, z \in \mathbb{Z}$  and assume  $x \sim y$ ,  $y \sim z$ . Then

$$(x - y)(x + y) = 7n \quad \text{while} \quad (y - z)(y + z) = 7m,$$

for some integers  $n$  and  $m$ . Alternatively,

$$7n = x^2 - y^2 \quad \text{and} \quad 7m = y^2 - z^2.$$

Add together and get

$$7(m + n) = x^2 - z^2 = (x - z)(x + z).$$

Hence  $(x - z)(x + z)$  is divisible by 7 and thus  $x \sim z$ . ■

**Example 5.2.6** *Not given* Let  $F$  be the set of fractions

$$F = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Define  $\sim$  on  $F$  by

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{if, and only if,} \quad ad = bc.$$

Show that  $\sim$  is an equivalence relation.

*Reflexive:* Starting from the fact that multiplication in integers is commutative we have

$$ab = ba \Rightarrow \frac{a}{b} \sim \frac{a}{b}.$$

*Symmetric* Again using the fact that multiplication in integers is commutative

$$\frac{a}{b} \sim \frac{c}{d} \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow \frac{c}{d} \sim \frac{a}{b}.$$

*Transitive*

$$\frac{a}{b} \sim \frac{c}{d} \Rightarrow ad = bc \quad \text{while} \quad \frac{c}{d} \sim \frac{e}{f} \Rightarrow cf = ed.$$

Multiply the first equality by  $f$  to get

$$adf = bcf = bed,$$

from the second equality. By the definition of  $F$  we have  $d \neq 0$  so we can divide by this to get  $af = be$ , which implies

$$\frac{a}{b} \sim \frac{e}{f}.$$

Thus  $\sim$  is an equivalence relation. ■

### 5.3 Partitions

**Definition 5.3.1** Let  $X$  be a non-empty set. A **partition** of  $X$  is a subset

$$\Pi \subseteq \mathcal{P}(X),$$

i.e. a collection of subsets, such that

i) the sets in  $\Pi$  are non-empty, so  $A \in \Pi \Rightarrow A \neq \emptyset$ ,

ii) the sets in  $\Pi$  are disjoint, so

$$\forall A_1, A_2 \in \Pi : A_1 \neq A_2 \Rightarrow A_1 \cap A_2 = \emptyset,$$

iii) the sets cover  $X$ , i.e.  $X = \bigcup_{A \in \Pi} A$ , or equivalently,

$$\forall x \in X, \exists A \in \Pi : x \in A.$$

We call the sets in  $\Pi$  the **parts** of the partition.

#### Examples

(a) Possible partitions of  $\mathbb{Z}$  are

(i)

$$\Pi = \{\{\text{odd integers}\}, \{\text{even integers}\}\} = \{[0]_2, [1]_2\} = \mathbb{Z}_2.$$

In general, for  $m \in \mathbb{N}$ ,  $\mathbb{Z}_m$  is a partition of  $\mathbb{Z}$ .

(ii)

$$\Pi = \{\{n \in \mathbb{Z} : n < 0\}, \{0\}, \{n \in \mathbb{Z} : n > 0\}\}.$$

(b) But

$$\{\{n \in \mathbb{Z} : n < 0\}, \{n \in \mathbb{Z} : n > 0\}\}$$

is **not** a partition of  $\mathbb{Z}$  since 0 is in no part.

Similarly

$$\{\{n \in \mathbb{Z} : n \leq 0\}, \{n \in \mathbb{Z} : n \geq 0\}\}$$

is **not** a partition of  $\mathbb{Z}$  since the parts are not disjoint.

(c) If  $A = \{a, b, c, d, e, f\}$  then

$$\Pi = \{\{a, b\}, \{c, e\}, \{d\}, \{f\}\}$$

is a partition of  $A$ .

## 5.4 From Relations to Partitions.

Just as we went from congruences (equivalence relations on  $\mathbb{Z}$ ) to congruence classes (a partition of  $\mathbb{Z}$ ) we can go from an equivalence relation to a partition.

**Definition 5.4.1** Suppose that  $\sim$  is an equivalence relation on a non-empty set  $X$ . For each  $a \in X$  define the **equivalence class of  $a$**  to be the set of elements of  $X$  related to  $a$ . Denote this class by  $[a]$  so

$$[a] = \{x \in X : x \sim a\}.$$

Alternative definitions are

$$[a] = \{x \in X : xRa\} = \{x \in X : (x, a) \in \mathcal{R}\}.$$

**Definition 5.4.2** The **Quotient Space** is the set of all equivalence classes

$$X/\sim = \{[a] : a \in X\}.$$

**Example 5.4.3** When  $X = \mathbb{Z}$  and  $a \sim b$  was  $a \equiv b \pmod{m}$  we wrote  $[a]_m$  in place of  $[a]$  and  $\mathbb{Z}_m$  in place of  $\mathbb{Z}/(\equiv \pmod{m})$ .

**Aside** What we managed to do for  $\mathbb{Z}_m$  was to define addition and multiplication on  $\mathbb{Z}_m$ , to give it some “arithmetic structure”. That would be the aim with other examples of  $X$  and  $\sim$ , but this is **not** achieved in this course. **End of aside**

**Question** Why do we demand that  $\sim$  is an *equivalence relation*? See the proof of the next result which uses all three defining properties of an equivalence relation.

**Theorem 5.4.4** Suppose that  $\sim$  is an equivalence relation on a non-empty set  $X$ . Then for  $a, b \in X$ ,

- i) If  $a \sim b$  then  $[a] = [b]$ ,
- ii) If  $a \not\sim b$  then  $[a] \cap [b] = \emptyset$ .

**Proof** p.267 I omitted the proof in lectures but said that it was a simple rewriting of the same result for congruence classes seen earlier. For this reason it **is** examinable.

(i) Assume  $a \sim b$ . To show that  $[a] = [b]$  we need show that  $[a] \subseteq [b]$  and  $[b] \subseteq [a]$ .

Let  $k \in [a]$ . By definition of equivalence class this means  $k \sim a$ . Thus we have both  $k \sim a$  and  $a \sim b$ . So by *transitivity*  $k \sim b$ . Hence  $k \in [b]$  by definition of the class. True **for all**  $k \in [a]$  means that  $[a] \subseteq [b]$ .

Let  $\ell \in [b]$ . By definition of the class this means  $\ell \sim b$ . Use *symmetry* to write  $a \sim b$  as  $b \sim a$ . Thus we have both  $\ell \sim b$  and  $b \sim a$ . So by *transitivity*  $\ell \sim a$ . Hence  $\ell \in [a]$  by definition of the class. True **for all**  $\ell \in [b]$  means  $[b] \subseteq [a]$ .

Combining  $[a] \subseteq [b]$  and  $[b] \subseteq [a]$  gives  $[a] = [b]$ .

(ii) Assume  $a \not\sim b$ . Assume for a contradiction that  $[a] \cap [b] \neq \emptyset$ , so there exists  $c \in [a] \cap [b]$ . From this we have both  $c \in [a]$  and  $c \in [b]$ .

From  $c \in [a]$  we get  $c \sim a$  while  $c \in [b]$  gives  $c \sim b$ . Use *symmetry* to write  $c \sim a$  as  $a \sim c$ . Thus we have both  $a \sim c$  and  $c \sim b$ . So, by *transitivity*,  $a \sim b$ . This contradicts the assumption that  $a \not\sim b$  thus the last assumption is false, i.e.  $[a] \cap [b] = \emptyset$ . ■

**Corollary 5.4.5** *Suppose that  $\sim$  is an equivalence relation on a non-empty set  $X$ . Then  $X/\sim$  is a partition on  $X$ .*

**Proof** One, and exactly one of  $a \sim b$  or  $a \not\sim b$  is true, so one and exactly one of  $[a] = [b]$  or  $[a] \cap [b] = \emptyset$  is true, i.e. equivalence classes are either disjoint or the identical. We never get partial intersection of classes. Every element is in some class (the class labeled by it) and thus every class is non-empty. Thus the collection of equivalence classes is a partition of  $X$ . ■

**Notation** Given  $X/\sim$  is a partition on  $X$  write  $\Pi_{\sim}$  for this partition, which we say is *induced* by  $\sim$ .

**Example 5.4.6** *Let  $X = \mathbb{Z}$  and  $\sim$  be given by  $x \sim y$  if, and only if,  $(x - y)(x + y)$  is divisible by 7. What do the equivalence classes look like?*

**Solution** Note that  $(x - y)(x + y) = x^2 - y^2$  so  $(x - y)(x + y)$  is divisible by 7 if, and only if,  $x^2 - y^2$  is divisible by 7.

Look at some classes. So

$$\begin{aligned} [1] &= \{x \in \mathbb{Z} : x \sim 1\} = \{x \in \mathbb{Z} : x^2 - 1 \text{ is divisible by } 7\} \\ &= \{x \in \mathbb{Z} : x^2 \equiv 1 \pmod{7}\} \\ &= \{x \in \mathbb{Z} : x \equiv 1 \text{ or } -1 \equiv 6 \pmod{7}\} \end{aligned}$$

Look at a class containing an element not in this list, i.e.

$$\begin{aligned} [2] &= \{x \in \mathbb{Z} : x^2 - 2^2 \text{ is divisible by } 7\} \\ &= \{x \in \mathbb{Z} : x^2 \equiv 4 \pmod{7}\} \\ &= \{x \in \mathbb{Z} : x \equiv 2 \text{ or } -2 \equiv 5 \pmod{7}\}. \end{aligned}$$

All the classes can be written in terms of congruence classes mod 7 as

$$[1] = [1]_7 \cup [6]_7, [2] = [2]_7 \cup [5]_7, [3] = [3]_7 \cup [4]_7 \text{ and } [0] = [0]_7.$$

■

**Example 5.4.7** *Not given* Let  $F$  be the set of fractions

$$F = \left\{ \frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

and  $\sim$  defined on  $F$  by

$$\frac{a}{b} \sim \frac{c}{d} \text{ if, and only if, } ad = bc.$$

What do the equivalence classes look like?

**Solution** A class is

$$\left[ \frac{a}{b} \right] = \left\{ \frac{c}{d} : c, d \in \mathbb{Z}, d \neq 0, ad = bc \right\}.$$

This class has many labels, but a special one is where  $a, b$  are coprime, i.e.  $\gcd(a, b) = 1$ . In fact if  $\gcd(a, b) = 1$  and

$$\frac{c}{d} \in \left[ \frac{a}{b} \right],$$

then  $cb = da$ . Since  $b|LHS$  then  $b|RHS$ , i.e.  $b|da$ . Yet  $\gcd(a, b) = 1$  with  $b|da$  implies  $b|d$ , i.e.  $d = bm$  for some  $m \in \mathbb{Z}$ . Substituting back in gives  $cb = bma$ , i.e.  $c = ma$ . Thus

$$\left[ \frac{a}{b} \right] = \left\{ \frac{ma}{mb} : m \in \mathbb{Z} \right\}.$$

For example

$$\left[ -\frac{5}{6} \right] = \left\{ \dots, \frac{10}{-6}, \frac{-5}{6}, \frac{5}{-6}, \frac{-10}{12}, \frac{15}{-18}, \dots \right\}.$$

We could then define the rational number  $-5/6$  to be this class. The set of classes, under this identification, would then be  $\mathbb{Q}$ . ■

## 5.5 From Partitions to Relations.

Partitions lead to relations in the following way.

**Definition 5.5.1** Given a partition  $\Pi$  of  $X$  define a relation  $\sim_{\Pi}$  by

$$\forall a, b \in X, a \sim_{\Pi} b, \text{ if, and only if, } \exists A \in \Pi : a, b \in A,$$

i.e.  $a$  and  $b$  lie in the same part of  $\Pi$ .

We say that  $\sim_{\Pi}$  is the partition induced by  $\Pi$ .

Alternatively, the relation can be defined as the set of ordered pairs

$$\begin{aligned} \mathcal{R}_{\Pi} &= \{(a, b) : a, b \in X, a \sim_{\Pi} b\} \\ &= \{(a, b) : \exists A \in \Pi \text{ with } a, b \in A\}. \end{aligned}$$

**Example 5.5.2** Let  $A = \{a, b, c, d, e, f\}$  and  $\Pi = \{\{a, c\}, \{b, d\}, \{e\}, \{f\}\}$  be a partition of  $A$ . Then

$$\begin{aligned} \mathcal{R}_{\Pi} &= \{(a, a), (a, c), (c, a), (c, c), (b, b), (b, d) \\ &\quad (b, d), (d, b), (d, d), (e, e), (f, f)\}. \end{aligned}$$

**Theorem 5.5.3** Let  $\Pi$  be a partition of  $X$  and  $\sim_{\Pi}$  the associated relation. Then  $\sim_{\Pi}$  is an equivalence relation.

**Proof** p.265 Let  $\Pi$  be a partition of  $X$ .

*Reflexive* Let  $a \in X$ . Then there exists  $A \in \Pi$  such that  $a \in A$ . It is then trivial to say that  $a, a \in A$  which is the definition of  $a \sim_{\Pi} a$ .

*Symmetric* Let  $a, b \in X$ . Assume  $a \sim_{\Pi} b$ . By definition of  $\sim_{\Pi}$  there exists  $A \in \Pi$  such that  $a, b \in A$ . It is then trivial to say that  $b, a \in A$  which is the definition of  $b \sim_{\Pi} a$ .

*Transitive* Let  $a, b, c \in X$ . Assume  $a \sim_{\Pi} b$  and  $b \sim_{\Pi} c$ . This means there exist  $A_1, A_2 \in \Pi$  such that  $a, b \in A_1$  and  $b, c \in A_2$ . Here  $b \in A_1$  and  $b \in A_2$  means that  $A_1 \cap A_2 \neq \phi$ . But by the definition of a partition if classes are not disjoint they are identical, so  $A_1 = A_2$  which we relabel as simply  $A$ . Thus  $a, b, c \in A$ . Here  $a, c \in A$  is the definition of  $a \sim_{\Pi} c$  as required. ■

**See the appendix** for details of starting with a partition  $\Pi$  on  $X$ , inducing a relation  $\sim_{\Pi}$  and then inducing a partition  $\Pi_{\sim_{\Pi}}$ . It can shown that  $\Pi_{\sim_{\Pi}} = \Pi$ , i.e. you return to the beginning.

Alternatively you can start with a relation  $\sim$  on  $X$ , induce a partition  $\Pi_{\sim}$  and continue to induce a relation  $\sim_{\Pi_{\sim}}$ . Again you return to the beginning since  $\sim_{\Pi_{\sim}} = \sim$ .